MATH3210 - SPRING 2024 - SECTION 004

HOMEWORK 10

Problem 1. Prove the divergence criteria of the comparison test. That is, show that if $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n \geq b_n \geq 0$ for every n, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Use the following scheme: Let $t_m = \sum_{n=1}^{m} b_n$ be the sequence of partial sums for b_n. Show that if $\sum_{n=1}^{\infty} b_n$ diverges, then t_m must diverge to infinity. Then show that the sequence t_m diverges to ∞, then so does $s_m = \sum_{n=1}^{m} a_n$.

Solution. We will first show that if $\sum_{n=1}^{\infty} b_n$ diverges, then the partial sums $t_m = \sum_{n=1}^{m} b_n$ diverge to infinity. Note that t_m is increasing since $b_n \geq 0$ for all n. Therefore, by the monotone convergence theorem and because t_m diverges, t_m is unbounded. Since it is increasing and unbounded, it diverges to ∞.

We claim that the sequence of partial sums $s_m = \sum_{n=1}^{\infty} a_n$ also diverges to infinity. Indeed, let $B > 0$. Since t_m diverges to ∞, there exists $M \in \mathbb{N}$ such that if $m \geq M$, $t_m > B$. Then if $m \geq M$, $s_m \geq t_m > B$, so s_m diverges to ∞. □

Problem 2. Determine whether the series converges or diverges. Prove that your answer is correct using the following tools only: the term test, the comparison test, the ratio test, and the integral test.

(a) $\displaystyle\sum_{n=1}^{\infty} \cos(n)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^8}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+3^n}{5^n}$

Solution.

(a) We claim that the series *diverges*. Indeed, since $\cos(n)$ does not converge to 0, it diverges by the term test.

(b) We claim that the series *converges*. Indeed, we use the comparison test with $b_n = \frac{1}{n^2}$. Notice that b_n converges since it is a p-series with $p > 1$, and that $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$. Therefore, by the comparison test, $\frac{1}{n^2+1}$ converges.

(c) We claim that the series *converges*. We apply the ratio test, which concludes convergence as long as $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Notice that

$$\lim_{n\to\infty} \frac{(n+1)^8}{(n+1)!} \cdot \frac{n!}{n^8} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^8 \cdot \frac{1}{n+1} = 1 \cdot 0 = 0$$

(d) We claim that the series *converges*. We apply the ratio test, which concludes convergence as long as $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Notice that

$$\lim_{n\to\infty} \frac{(n+1)^3+3^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n^3+3^n} = \lim_{n\to\infty} \frac{(n+1)^3/3^n+3}{5(n^3/3^n+1)} = \frac{0+3}{5(0+1)} = 3/5.$$

□

Problem 3. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of positive functions such that $\sum_{n=1}^{\infty} f_n(0)$ converges, and each f_n is L_n-Lipschitz. Show that if $L = \sum_{n=1}^{\infty} L_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to an L-Lipschitz function.

Solution. We claim that for every $x \in [a, b]$, the sequence $\sum_{n=1}^{m} f_n(x)$ is Cauchy. Indeed, fix $\varepsilon > 0$. Choose an N such that $\sum_{n=N}^{\infty} L_n < \dfrac{\varepsilon}{2(b-a)}$ and $\sum_{n=N}^{\infty} f_n(a) < \varepsilon/2$. Such an N exists since the sequence of partial sums is Cauchy for each sequence. Then if $m_2, m_1 \geq N$,

$$\left|\sum_{n=1}^{m_2} f_n(x) - \sum_{n=1}^{m_1} f_n(x)\right| \leq \left|\sum_{n=m_1+1}^{m_2} f_n(a) + f_n(x) - f_n(a)\right| \leq \sum_{n=m_1+1}^{m_2} |f_n(a)| + |f_n(x) - f_n(a)|$$

$$\leq \sum_{n=m_1+1}^{\infty} |f_n(a)| + \sum_{n=m_1+1}^{\infty} L_n |x-a| < \varepsilon/2 + \left|\frac{x-a}{b-a}\right| \cdot \varepsilon/2 < \varepsilon.$$

Thus, by the topological completeness theorem, for each x, the series $\sum_{n=1}^{\infty} f_n(x)$ converges. Since the choice of N does not depend on x for the Cauchy property, the functions f_n coverge uniformly to a function f. Furthermore,

$$|f(x) - f(y)| = \left|\sum_{n=1}^{\infty} f_n(x) - f_n(y)\right| \leq \sum_{n=1}^{\infty} |f_n(x) - f_n(y)| \leq \sum_{n=1}^{\infty} L_n |x-y| = L|x-y|.$$

So f is L-Lipschitz. □

Problem 4. Let $f(x) = xe^x$. Find a number N such that the Taylor approximation of order N is within .1 of $f(x)$ on the interval $[0, 1]$. [*Hint*: First, find and prove a formula for $f^{(k)}(x)$ by induction, then bound $f^{(k)}(x)$ on $[0, 1]$ using this formula by a number depending on k]

Solution. We first claim that $f^{(k)}(x) = ke^x + xe^x$. we show it by induction. It holds for $k = 0$, since $f^{(0)}(x) = f(x) = 0 \cdot e^x + xe^x$. If $f^{(k)}(x) = ke^x + xe^x$, then

$$f^{(k+1)}(x) = (ke^x + xe^x)' = ke^x + (e^x + xe^x) = (k+1)e^x + xe^x.$$

It follows that on $[0, 1]$, $f^{(k+1)}x) \leq (k+1)e$. Hence, the remainder term of the Nth Taylor polynomial is

$$|R_N(x)| = \frac{\left|f^{(N+1)}(c)\right|}{(N+1)!} \cdot |x-0|^{N+1} \leq \frac{(N+1)e}{(N+1)!} = \frac{e}{N!}$$

Taking $N = 5$ yields that $e/5! < 3/(5 \cdot 4 \cdot 3 \cdot 2) = \frac{1}{40} < 1/10$, so the 5th Taylor approximation is within .1 on $[0, 1]$. □